Modeling of crop stem deflection in the context of combine harvester reel design and operation

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Abstract: A model of crop stem deflection by the combine harvester reel is formulated. The equations derived thereof are evaluated on the basis of empirical data that were acquired through deflection of crop stems in a ready-for-harvest *Japonica* rice in the field. The empirical data are found to be in agreement with the theoretically derived equations. Applications of crop stem deflection to reel design and operation are discussed. The derived crop stem deflection model should be applicable in other situations in which it becomes necessary to study the deflection of crop stems, particularly in the domain of agricultural machinery engineering.

Keywords: combine harvester header, tined reel, crop stem deflection, rice crop, mathematical modeling, model validation, model application

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1 Introduction

Between the moment that the reel first engages uncut crop stems and the moment that the stems so engaged are cut by the cutterbar, the crop stems undergo deflection under the action of combine harvester reel slats or tine bars. This phenomenon, illustrated in Figure 1, should have important implications on the design and operation of the combine harvester reel.

An earlier effort to study the interaction between the combine harvester reel and the crop was largely experimental, without attempting to model crop stem deflection mathematically (Quick, 1973). In this paper, the objective is to understand crop stem deflection characteristics, under the action of the combine harvester reel, and to apply these characteristics in combine harvester reel design and operation. A model of the deflection of crop stems by the combine harvester reel is formulated. The model is evaluated on the basis of empirical data acquired through deflection of the stems of a ready-for-harvest *Japonica* rice crop in the field. Applications of crop stem deflection characteristics, in combination with reel kinematics, for the determination of reel stagger, as well as the estimation of the appropriate number of reel slats or tine bars, are presented and discussed.

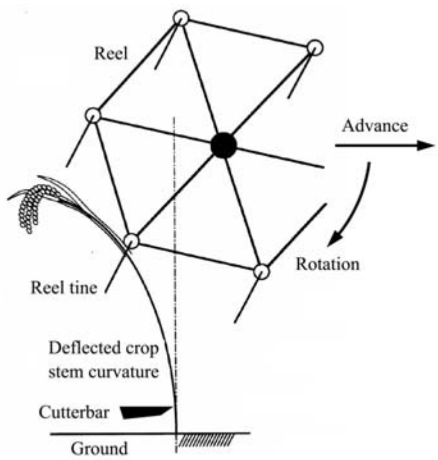

Figure 1 Deflection of an uncut stem by the reel

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2 Model formulation

2.1 Assumptions

1) A bunch of stems deflected by the reel shall be considered to behave like a single, initially vertical cantilever that is fixed at the base.

2) At its point of action, the deflecting force shall be considered to be directed normal to the curvature of the cantilever.

3) The stress-strain relationship for the deflected stems shall be assumed to be linearly elastic.

The assumption of small deflections, commonly made in mechanical and structural engineering, is not made here. Referring to Figure 2a, the coordinate system has been chosen so as to realize a right-handed system with positive deflection directed from right to left. According to the elementary theory of elastic bending (Den Hartog, 1977; Popov, 1990; Hibbeler, 2008), at an arbitrary point, denoted (Y, Z), along the length of the deflected cantilever, we may write:

$$\frac{1}{r}=\frac{\mathrm{d}\phi}{\mathrm{d}s}=\frac{M}{EI} \tag{1}$$

Furthermore, it should be evident in Figure 2b that the bending moment at the point (Y, Z), is:

$$M=F[(Y_m-Y)\cos\phi_m+(Z_m-Z)\sin\phi_m] \tag{2}$$

where, Y_m and Z_m are the coordinates of the point of contact between the force F and the crop stem.

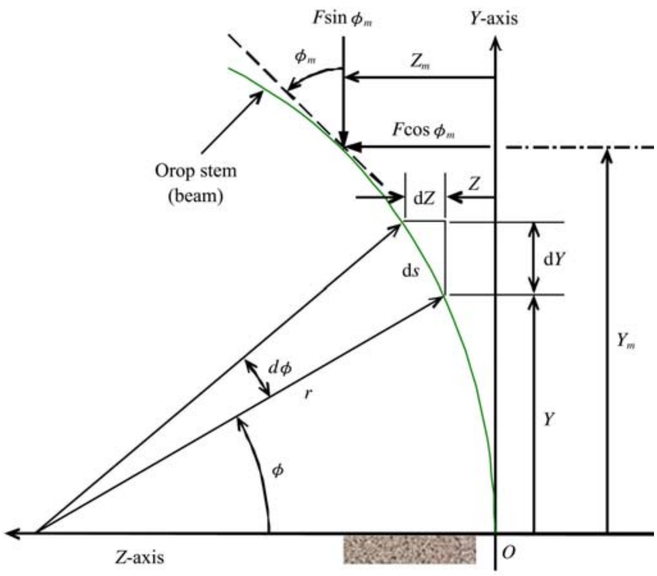

Figure 2a Deflection of the stem due to external force F

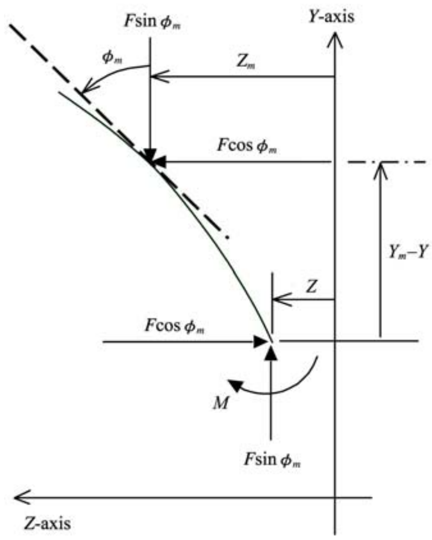

Figure 2b Free body diagram of a segment of the deflected stem

Figure 3a illustrates the deflection model with the coordinates transformed in a manner that is similar, but not equivalent, to the transformation from rectangular Cartesian coordinates to polar coordinates, which is common in the study of kinematics (Tuma, 1974). Referring to Figure 3b, this transformation is obtained as follows:

$$\left.\begin{aligned}L_m-L&=bc+cf=(Y_m-Y)\cos\phi_m+(Z_m-Z)\sin\phi_m\\ N_m-N&=ac-ad=(Y_m-Y)\sin\phi_m-(Z_m-Z)\cos\phi_m\end{aligned}\right\} \tag{3a}$$

Equation (3a) can be readily combined into a single matrix equation, as follows:

$$\begin{Bmatrix}(L_m-L)\\(N_m-N)\end{Bmatrix}=\begin{bmatrix}\cos\phi_m & \sin\phi_m\\ \sin\phi_m & -\cos\phi_m\end{bmatrix}\begin{Bmatrix}(Y_m-Y)\\(Z_m-Z)\end{Bmatrix} \tag{3b}$$

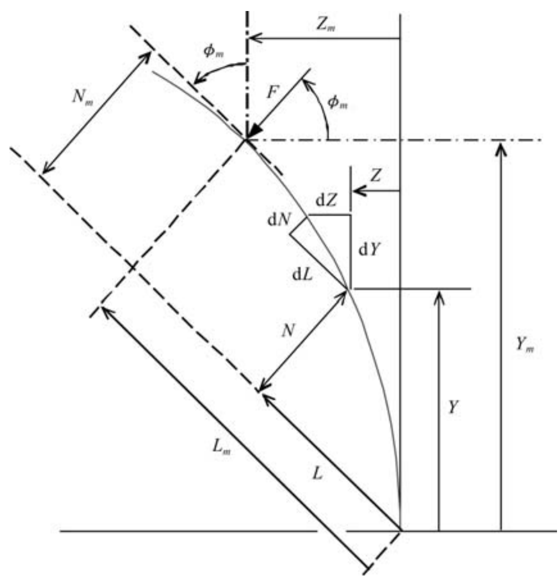

Figure 3a Transformed model of a deflected stem

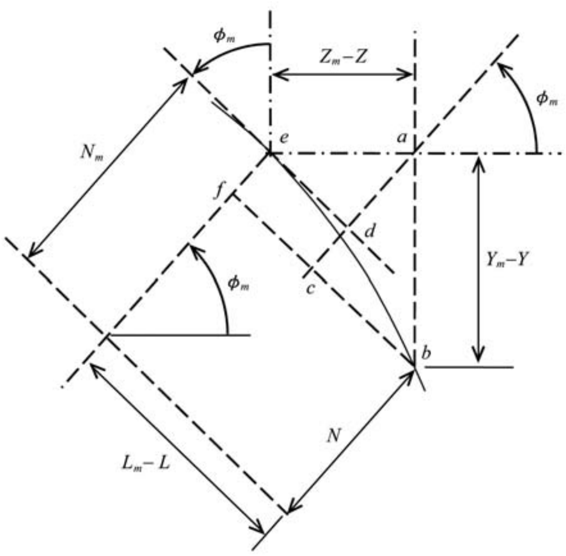

Figure 3b Construction for the derivation of Equation (3a)

In Equation (3b), since L and N are both zero when Y and Z are both zero, it follows that:

$$\begin{Bmatrix} L_m \\ N_m \end{Bmatrix} = \begin{bmatrix} \cos\phi_m & \sin\phi_m \\ \sin\phi_m & -\cos\phi_m \end{bmatrix} \begin{Bmatrix} Y_m \\ Z_m \end{Bmatrix} \tag{3c}$$

The transformation matrix of Equation (3b) and (3c) is a special one. It is equal to its own transpose and therefore symmetric. Its transpose is also equal to its inverse and therefore the matrix is orthogonal, as can be seen in Equation (3d) below. This also means that the matrix is equal to its own inverse. Such a matrix is also known as an involutory matrix (Sawyer, 1982).

$$\begin{bmatrix} \cos\phi_m & \sin\phi_m \\ \sin\phi_m & -\cos\phi_m \end{bmatrix} \begin{bmatrix} \cos\phi_m & \sin\phi_m \\ \sin\phi_m & -\cos\phi_m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3d}$$

In the context of plane transformation geometry, this matrix has the form of a reflection matrix (Sawyer, 1982), that maps (Y, Z) coordinate into their mirror image (L, N). Therefore:

$$L_m = Y_m \quad \text{and} \quad N_m = Z_m \tag{3e}$$

The relationships in Equation (3e) may not be apparent by looking a Figure 3a. In any case, that figure is not drawn to scale. However, the validity of Equation (3e) can be checked by substituting them into Equation (3c) to obtain the following:

$$\left.\begin{aligned} Y_m &= Y_m\cos\phi_m + Z_m\sin\phi_m \\ Z_m &= Y_m\sin\phi_m - Z_m\cos\phi_m \end{aligned}\right\} \tag{3f}$$

By squaring each of the equations in (3f), the following is readily obtained:

$$\left.\begin{aligned} Y_m^{\ 2} &= Y_m^{\ 2}\cos^2\phi_m + 2Y_mZ_m\sin\phi_m\cos\phi_m + Z_m^{\ 2}\sin^2\phi_m \\ Z_m^{\ 2} &= Y_m^{\ 2}\sin^2\phi_m - 2Y_mZ_m\sin\phi_m\cos\phi_m + Z_m^{\ 2}\cos^2\phi_m \end{aligned}\right\} \tag{3g}$$

Finally, by adding corresponding terms in Equation (3g), the following is obtained, which completes the check for validity of Equation (3e):

$$Y_m^{\ 2} + Z_m^{\ 2} = Y_m^{\ 2} + Z_m^{\ 2} \tag{3h}$$

With the results obtained in Equation (3a) through (3h), it follows that, in Figure 3a, the straight line joining the origin O of the coordinate system to the point of contact between the external force F and the crop stem, makes an angle of $\phi_m/2$ with the vertical Y axis. Therefore:

$$\phi_m = 2\tan^{-1}\left(\frac{Z_m}{Y_m}\right) \tag{4}$$

Equation (4) is based entirely on the geometry of deflection. It does not involve material properties of the deflected cantilever. Therefore, Equation (4) can be used whether the material of the deflected beam is purely elastic or otherwise.

Moreover, from Equation (2), (3a) and in Figure 3c, it is evident that the bending moment at the point (Y, Z) on the deflected stem, due to the externally applied force F, is given by:

$$M = F(L_m - L) \tag{5}$$

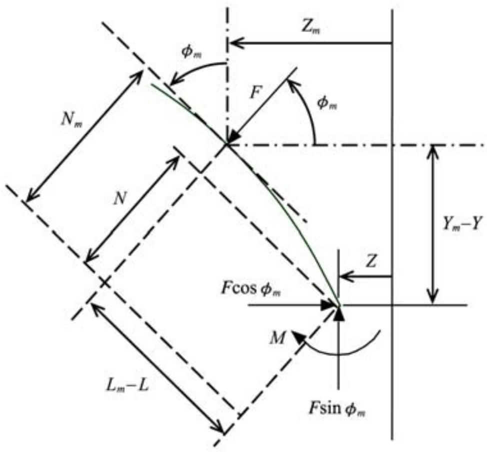

Figure 3c Free body diagram of a segment of the deflected stem

Figure 3d, which is extracted from Figure 2a and 3a, shows an enlarged stem element at the point (Y, Z) along the length of the deflected stem. It can be seen in Figure 3d that:

$$\frac{\mathrm{d}L}{\mathrm{d}s} = \cos(\phi_m - \phi) \tag{6}$$

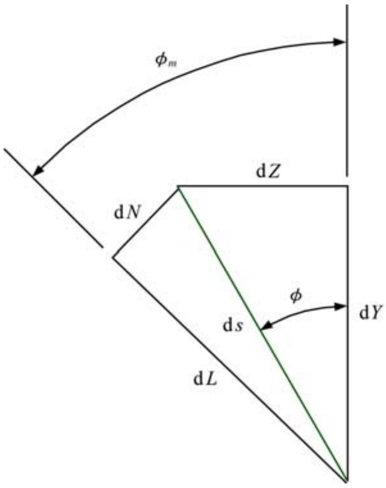

Figure 3d Enlarged stem element

From Equation (1), (5) and (6), it follows that:

$$\cos(\phi_m - \varphi) = \frac{\mathrm{d}\phi}{\mathrm{d}s} \times \frac{\mathrm{d}L}{\mathrm{d}\phi} = \frac{M}{EI} \times \frac{\mathrm{d}L}{\mathrm{d}\phi} = \frac{F}{EI}(L_m - L) \times \frac{\mathrm{d}L}{\mathrm{d}\phi}$$

Therefore:

$$\cos(\phi_m - \phi) = \frac{F}{EI}(L_m - L)\mathrm{d}L \tag{7}$$

Timoshenko and Gere (1961) formulated a similar model, involving large lateral deflection of a column, which led to an elliptical integral. Elliptical integrals can only be evaluated numerically. In contrast, the introduction here of L as the variable of integration instead of s greatly simplifies the problem.

Equation (7) can be readily integrated provided that the relationship between EI and L is known. The following possible scenarios will be considered, compared and contrasted:

1) The product EI, which is a measure of the stems' ability to resist flexural deformation, does not vary with L. Since $\phi = 0$ when L=0, Equation (7) can then be integrated to obtain the following:

$$\sin(\phi_m) = \frac{F}{2EI} L_m^{\ 2} \tag{8}$$

2) The stems are naturally so formed as to be of uniform strength throughout their lengths if they are loaded as cantilevers with concentrated loads at or near their free ends. Suggs and Splinter (1965) made a similar assumption in their study of the mechanical properties of tobacco stalks.

Here, strength refers to the ability of the stems to withstand flexural loading. Thus, uniform strength would require that, under the prescribed loading, the flexural (bending) stress does not vary along the length of the stems.

The bending stress at the surface of the stalk, at any cross section along its loaded length, may be expressed as follows, if the cross section is assumed to be circular with a diametral neutral plane (Hearn, 1985):

$$\sigma = \frac{32M}{\pi D^3} \tag{9a}$$

where, D is the diameter of the cross section.

Thus, from Equation (5) and (9a), the bending stress at the cross section at the base of the stem (L=0, diameter D_0) would be:

$$\sigma_0 = \frac{32FL_m}{\pi D_0^{\ 3}} \tag{9b}$$

Similarly, for an arbitrary cross section (arbitrary L, diameter D):

$$\sigma = \frac{32F(L_m - L)}{\pi D^3} \tag{9c}$$

If the stress is to remain constant throughout the length of the stem then σ may be equated to σ_0, leading to the following result:

$$\frac{D_0}{D} = \left[\frac{L}{L_m - L}\right]^{\frac{1}{3}} \tag{9d}$$

The second moment of area about a diameter of a circular cross section is given by (Popov, 1990):

$$I = \frac{\pi D^4}{64} \tag{9e}$$

Denoting the second moment of area at the base of the stem by I_0 and using Equations (9d) and (9e) leads to the following result:

$$\frac{I_0}{I} = \left[\frac{D_0}{D}\right]^4 = \left[\frac{L}{L_m - L}\right]^{\frac{4}{3}} \tag{9f}$$

Further, along with Equation (9f), if the modulus of elasticity, E, is assumed to be constant, then Equation (7) may be integrated to obtain the following:

$$\sin(\phi_m) = \frac{3FL_m^{\ 2}}{2EI_0} \tag{9g}$$

Apparently the nature of variation of EI with L may not alter the essential form of the equation corresponding to Equation (8) or (9g). In both equations, $\sin(\phi_m)$ is proportional to $L^2{}_m$.

More generally, we assume that EI varies with L in an unknown but continuous manner that may therefore be represented in the form of a Taylor-McLaurin polynomial (Larson, Hostetler and Edwards, 1994; Zill, 1985) as follows:

$$\frac{1}{EI}=\frac{1}{E_0I_0}\left[1+\sum_{i=1}^{k}A_iL^i\right] \tag{10}$$

where, E_0I_0 corresponds to the base of the cantilever and (k+1) is the number of terms in the Taylor-McLaurin polynomial. Thus substituting into Equation (7) and integrating yields:

$$\int_0^{\varphi_m}\cos(\phi_m-\phi)d\phi=\frac{F}{E_0I_0}\int_0^{L_m}(L_m-L)\left[1+\sum_{i=1}^{k}A_iL^i\right]dL \tag{11}$$

As a result of which:

$$\sin(\phi_m)=\left[1+\sum_{i=1}^{k}\left[\left[\frac{2}{(i+1)(i+2)}\right]A_iL_m{}^i\right]\right]\frac{FL_m{}^2}{2E_0I_0} \tag{12a}$$

In Equation (12a), let us introduce the following notation:

$$B_i=\sum_{i=1}^{k}\left[\left[\frac{2}{(i+1)(i+2)}\right]A_iL_m{}^i\right] \tag{12b}$$

Then it follows that for i=0:

$$B_0=A_0 \tag{12c}$$

In the above expression, B_0 is a constant coefficient that should take on an appropriate value that agrees with Equation (12a). If B_0=1 then Equation (12a) can be re-written as follows:

$$\sin(\phi_m)=\left[\sum_{i=0}^{k}B_i\right]\frac{FL_m{}^2}{2E_0I_0} \tag{12d}$$

Equation (12d) will be used with empirical data to evaluate the model, as detailed in section 3 below.

3 Evaluation of the model

Measurement of deflection of the stems of a *Japonica* rice crop in the field has been described and discussed elsewhere (Oduori, 1994; Sakai, Inoue, and Oduori, 1993a). Relevant deflection data, Z_m, were sampled for three different rates of continuous deflection, U_d, with the deflecting force applied at three different heights, Y_m. For each run of the experiment, values of Y_m and U_d were set and then the corresponding values of Z_m were measured. Each set of treatments were replicated three times. The results are reviewed here by way of evaluation of the model formulated in the preceding section.

3.1 The relationship involving ϕ_m, X_m, and Y_m

Experimentally obtained values of ϕ_m were plotted against $\tan^{-1}(Z_m/Y_m)$ in order to study the validity of Equation (4). A typical plot is shown in Figure 4 for the case of Y_m = 0.45 and U_d = 0.01. In practice, U_d corresponds to the rate of deflection of the crop stems by the reel tine bar, which can be quite slow, being the horizontal component of the resultant of header advance velocity and tine bar peripheral velocity. At some point in the cycle of reel motion, this horizontal component of the resultant velocity actually becomes zero.

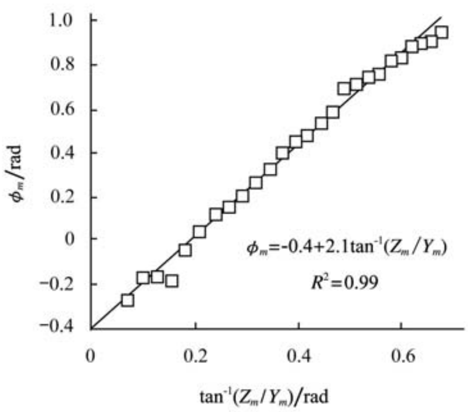

Figure 4 Relationship involving Z_m, Y_m and ϕ_m

In Figure 4, a least-squares regression line of the form $\phi_m=\beta_0+\beta_1\tan^{-1}(Z_m/Y_m)$ was fitted to the empirical data. Results of the linear regression for all nine treatments are given in Table 1.

Table 1 Results of the least-squares linear regression of ϕ_m on $\tan^{-1}(Z_m/Y_m)$

Height /m	Speed /m·s^{-1}	Y Intercept /rad	Gradient, β_1 (dimensionless)	Coefficient of determination, R^2
0.35	0.005	-0.12	1.52	0.99
0.35	0.010	-0.06	1.2	0.98
0.35	0.015	-0.265	1.698	0.992
0.40	0.005	-0.322	2.170	0.998
0.40	0.010	-0.154	1.693	0.945
0.40	0.015	-0.294	2.164	0.970
0.45	0.005	-0.096	1.781	0.938
0.45	0.010	-0.388	2.114	0.992
0.45	0.015	-0.354	1.996	0.985

Using average values of the data in Table 1, we may write:

$$\phi_m = -0.228 + 1.814\tan^{-1}\left(\frac{Z_m}{Y_m}\right) \tag{13}$$

Which appears to be in reasonable agreement with Equation (4), but in a few treatments, for instance, in row two, the gradient is "1.2" which is about half the predicted value of "2". This values is an outlier, though some natural variability can be expected in the behavior of natural entities such as crop stems.

With reference to Table 1, results of *t*-tests indicated that the mean of the empirical value of β_1 was not significantly different from the theoretical value (β_1=2 of Equation (4)). On the other hand, the mean value of β_0 was found to be significantly different from zero; therefore the assumption of an initially vertical cantilever is not quite valid, probably, to a large extent, due to the spreading out of crop foliage towards the top. Though β_1 was found to be moderately correlated to Y_m, a one-way ANOVA indicated that the effect of Y_m on β_1 was not significant even at the 5% level.

Let us now introduce the following quantity:

$$f(L_m) = \frac{\sin(\phi_m)}{FL_m^{\ 2}} \tag{13a}$$

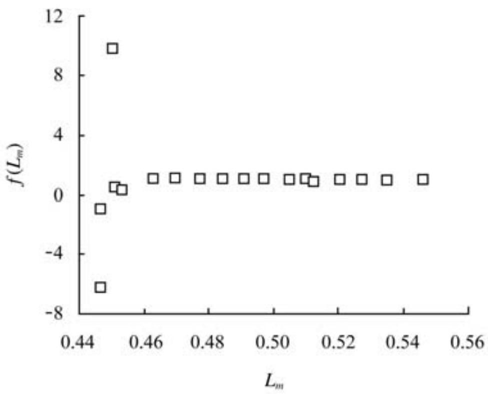

Figure 5 Example of a graph of $f(L_m)$ against L_m

In Figure 5, the quantity $f(L_m)$ was plotted against L_m, with the values of F, L_m and ϕ_m having been calculated from experimentally determined data. It is evident that except for the initial stages of deflection of the stems, the quantity denoted $f(L_m)$ remains substantially constant. This result implies that we may write:

$$\sin(\phi_m) = CFL_m^{\ 2} \tag{13b}$$

In Equation (13b), C is a constant and the equation has the general form suggested by the special cases of Equation (8) and (9g) and even (12d). The situation in the initial stages of deflection is probably complicated by the outward spread of the crop foliage as opposed to the idealized vertical cantilever model, but then it occupies only a small portion of the total deflection of the stems that is soon overtaken during the cycle of gathering of the crops by the reel.

It appears that the field data are in good agreement with the model, except for the initial stages of deflection. The equations derived from the model, which led to Equation (4) and Equation (13b), may therefore be tentatively applied to reel design and operation.

4 Applications

4.1 Determination of reel stagger

Application of crop stem deflection for the determination of combine harvester reel stagger, denoted Z_r in Figure 6, has been presented and discussed elsewhere. The underlying postulate is that crop stems should be cut by the cutterbar at the moment when, or before, the velocity of the ascending deflecting tine bar becomes tangential to the curvature of the deflected stems. As a consequence, the following equation was derived (Oduori, 1994; Sakai, Inoue, and Oduori, 1993a; Oduori, Sakai, and Inoue, 1993b; Oduori, Sakai, and Inoue, 1993, see Figure 7):

$$\omega t_m = \cos^{-1}\left(\frac{R_0}{R}\cos\phi_m\right) + \pi + \phi_m \tag{14}$$

This equation shall be seen to have a central role in the determination of reel stagger.

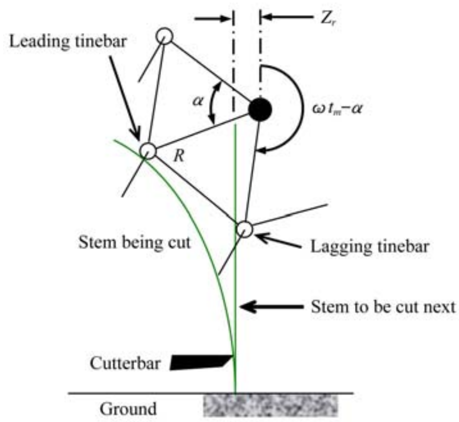

Figure 6 Relative positions of two successive tine bars

In Figure 7, the vertical distance labelled R_0 is equal, in magnitude, to the reel advance per radian of rotation of the reel.

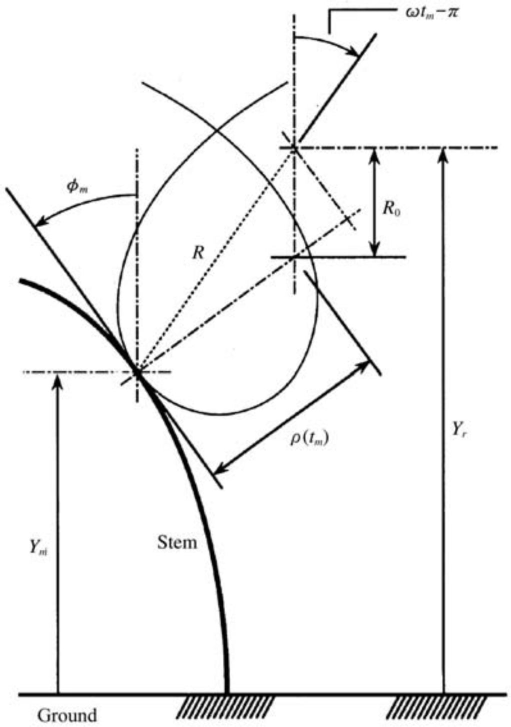

Figure 7 Relationship between stem deflection and reel kinematic parameters

4.2 Estimation of the appropriate number of tine bars on the reel

Another possible application of the deflection of crop stems by the reel is in the estimation of the appropriate number, n, of tine bars that the reel should have. The principle is illustrated in Figure 6. The rationale is that as the stems deflected by the leading tine bar are cut the lagging tine bar should be in such a position as to support the stems that will be cut next; otherwise the possibility arises of the crop that is cut without being supported by the reel eventually being lost by falling to the ground ahead of header. In practice some stems are sandwiched between those deflected by the leading tine bar and those deflected by the lagging tine bar, but the effect of these, on the equations to be derived concerning the number of tine bars on the reel, has been neglected.

In Figure 6, it can be shown that:

$$\alpha = \omega t_m - \pi - \sin^{-1}\left(\frac{Z_r}{R}\right) \tag{15a}$$

Now, from Equation (14) and (15a), the following can be obtained:

$$\alpha = \cos^{-1}\left(\frac{R_0}{R}\cos\phi_m\right) + \phi_m - \sin^{-1}\left(\frac{Z_r}{R}\right) \tag{15b}$$

Furthermore, the number of tine bars on the reel can be calculated as follows:

$$n = \frac{2\pi}{\alpha} \tag{16}$$

The following points should be noted with regard to Equation (15b) and (16) above:

1) The interaction of stems sandwiched between the leading and lagging tine bars has been neglected as mentioned above.

2) Both ϕ_m and Z_r vary with the type and condition of the crop. Therefore, the usefulness of Equation (15b) and (16) needs to be supported and validated by further research and acquisition of adequate empirical data.

3) Since n must be an integer, at best its calculated value is likely to be only an estimate.

Perhaps the best way to utilize Equation (14), (15a), (15b) and (16) would be to start with a known value of the number of tine bars on the reel, n, use Equation (16) to determine α and then useEquation (14) and (15a) to determine the reel stagger, Z_r.

5 Conclusions

1) A model of the deflection of a crop stem, as caused by the reel of a combine harvester, was developed.

2) Mathematical relations derived from the model have been evaluated against empirical data that were acquired through field measurement of the deflection of the stems of a *Japonica* rice crop. The model has been found to be in reasonable agreement with the empirical data, at least for the relevant experimental conditions.

3) The model, together with the equations of reel kinematics, have been applied for the determination of reel stagger. Because the calculated values depend on parameters that vary with the type and condition of the crop, their usefulness for commercial machine design purposes is unknown at the present time.

4) More extensive studies on crop stem deflection by the combine harvester reel are desirable. Particularly, studies involving more crops and wider ranges of the experimental conditions should be informative.

Nomenclature

A_i, B_i constant coefficients (i = 0, 1, 2, ...)

C constant coefficient

E modulus of elasticity, N m^{-2}

F deflecting force, N

I second moment of area, m^4

L, N coordinates perpendicular to, and parallel to the deflecting force, respectively, m

m subscript implying maximum

M bending moment, N m

n appropriate number of tinebars on the reel

r radius of curvature, m

R radius of reel, m

R_0 distance whose magnitude is equal to header advance per radian of reel rotation, m

s length along the deflected cantilever, m

t time, s

Y, Z Cartesian coordinates in the plane, m

Y_r height of the axis of the reel above the ground, m

Z deflection of the cantilever, m

Z_r reel stagger, m

α angular displacement between successive tine bars, rad

β_i regression coefficients (i = 0, 1, 2, ...)

ϕ angular deformation of the deflected cantilever , rad

$\rho(t)$ position vector of the tine bar relative to the instantaneous centre of rotation, m

ω rotational velocity of reel, rad s^{-1}

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