

Analytical and numerical solutions for spacing between bi-level drains with evapotranspiration

Ashutosh Upadhyaya¹, Manisha Maheshkumar Kankarej², Pawan Jeet^{1*}

(1. ICAR Research Complex for Eastern Region, Patna – 800 014, Bihar, India;

2. Rochester Institute of Technology, Dubai, United Arab Emirates)

Abstract: In this study, the spatial and temporal variation of water table between two bi-level drains has been investigated, through a numerical as well as analytical solution (AS) model, by solving one dimensional nonlinear Boussinesq equation incorporating depth dependent evapotranspiration (ET) term employing Hybrid finite analytic solution (HFAS), Galerkin's method based finite element solution (FES) and fully implicit finite difference solution (FIFDS). The spacing between the two drains was also determined by adopting a drainage criterion of a 30 cm falling in the maximum water table within 2-days from an initially flat water table close to the soil surface. Considering HFAS as the benchmark solution, the falling water tables obtained from FES, followed by FIFDS, were found to be in close agreement with HFAS. Based on L2 and Tchebycheff norm computations, indicates to average and maximum difference from the benchmark solution, FES and FIFDS were ranked first and second, respectively. The AS predicted a slower falling of the water table compared to HFAS and was therefore rated lower than the numerical solutions of the nonlinear Boussinesq equation. The drain spacing estimated using HFAS exceeded that obtained from the analytical solution by 14.32% for zero ET and 16.50% for depth-dependent ET. In comparison with FIFDS, the HFAS-derived spacing values were higher by 4.18% and 4.66%, while they exceeded those from FES by 4.06% and 4.58% for zero ET and depth-dependent ET, respectively. Owing to simplicity and ease of implementation of HFAS compared to other numerical models, it may be effectively employed for analyzing falling water table and computing drain spacing.

Keywords: bi-level drainage, Boussinesq equation, drain spacing, subsurface drainage, water table fall

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1 Introduction

Subsurface drainage is one of the most viable solutions to drain surplus water and harmful salts from the crop root zone and its main objective is lowering of water table (WT) in the root zone either for ameliorating aeration or reducing salinity. Crop-water-soil hydrological characteristics, crop considerations, depth to impermeable layer in the subsoil, extent of salinity control, accessibility of gravity outlet as well

as the capacity of drain laying equipment are the major factors that help in deciding the depth of subsurface drains (Ritzema, 2014). Considering the various factors which affect drain depth, spacing between drains is decided on the basis of a drainage criterion. Closer drain spacing increases the cost, which makes drainage systems uneconomical, and unreasonable (Chahar and Vadodaria, 2008). High initial investment in installation of subsurface drainage is one of the major constraints. Bi-level drainage system is the one in which by having deep and shallow drain lines on

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***Corresponding author: Pawan Jeet.** ICAR Research Complex for Eastern Region, Patna, Bihar, India. Email: pawan.btag@gmail.com.

alternate basis the excavation and digging work reduces substantially, and thus becomes economical as compared to conventional level drainage system (Verma et al., 1998; Hamzeh et al., 2010; Maddahzadeh et al., 2017). The other way of economizing the installation cost is consideration of ET from the fields. It plays an important role in drainage design in all the regions but in semi-arid and arid regions it assumes a special importance due to its significant effect on the lowering of water table. Arid and semi-arid regions are marked by harsh environmental conditions, including low and highly variable rainfall, strong winds, intense solar radiation, and high potential evapotranspiration throughout the year. Evapotranspiration (ET) is essential for effective irrigation planning, reliable assessment of crop productivity, and equitable water distribution, particularly in semi-arid regions (Singh et al., 2024).

DeBoer and Chu (1975) developed unsteady state theory for bi-level drainage system using Bouwer and Van Schilfgaarde (1963) approach. They suggested the use of a bi-level drainage system for having alternating deep and shallow drain lines. Henceforth, bi-level drains excavation can be minimized substantially in semi-arid and arid regions, which can save a lot of water in the irrigated field conditions, based on Dupuit–Forchheimer (D-F) assumptions. Chu and DeBoer (1976) validated the bi-level drainage theory with the field and laboratory experiments. The findings of this study indicated that the theory was adequate to explain the lowering of WT under a bi-level drainage condition. Sabti (1989, 1994) proposed an AS of linearized Boussinesq equation (BE) using method of Fourier transformation to predict transient movement of WT and also developed a numerical solution of nonlinear BE using backward difference principle. Verma et al. (1998) obtained AS to linearized Boussinesq boundary value problem involving Laplace transformation. Upadhyaya and Chauhan (2000) studied AS for linearized BE incorporating the effect of ET with appropriate initial and boundary conditions (IBC). Cook and Rassam (2002) proposed a 1-D analytical model for WT dynamics coupled with

evaporation and explained the importance of incorporating the effect of evaporation in the drainage designs. Verma et al. (2007) obtained AS for subsurface drainage boundary value problem using inversion theorem. Upadhyaya (2013) studied HFAS to calculate spacing between the drains in sloping lands.

Russell and Bedrikovetsky (2023), Sarmah et al. (2024), Upadhyaya et al. (2023), Verma et al. (2007) and Jain et al. (1994) studied one dimensional (1D) linearized BE with appropriate IBC has been solved analytically or 1D nonlinear BE has been solved using backward difference principle to study fall of WT in bi-level drainage system. Upadhyaya and Kankarej (2022) have developed two AS of linearized BE and three numerical solutions of nonlinear 1D BE to describe WT rise between two canals due to canal seepage (Tian et al., 2022), and time varying recharge. Since numerical solutions of 1D nonlinear BE incorporating depth dependent ET to describe WT fall and to compute spacing between bi-level drains have not been developed and impact of ET on spacing has not been studied, appropriate boundary conditions for bi-level drainage system was formulated. It was solved analytically and numerically and compared between lowering of WT and spacing between bi-level drains. Upadhyaya and Chauhan (2001) reported that HFAS, where the nonlinear BE is locally linearized and solved analytically after approximating unsteady term by a simple finite difference formula to approximately preserve nonlinear effect by assembly of locally analytic solutions, predicted midpoint WT between two drains in a horizontal aquifer quite close to the existing experimental results. Therefore, HFAS was considered as benchmark solution. Therefore, the objective of the present study is to compute WT profiles as well as spacing between bi-level drains obtained from all the solutions and compares it with those obtained from HFAS.

2 Materials and methods

2.1 Problem definition

Figure 1 illustrates conventional level and bi-level drainage problem. Here, initially WT is supposed to be horizontal and close to land surface at a height (h_0) above deep drains and shallow drain lying at a height (h_1) above deep drain, separated by a distance (L). The

aquifer is assumed homogeneous and isotropic. Deep drain has been considered as the origin. Accordingly, axis is parallel and perpendicular to impermeable barrier.

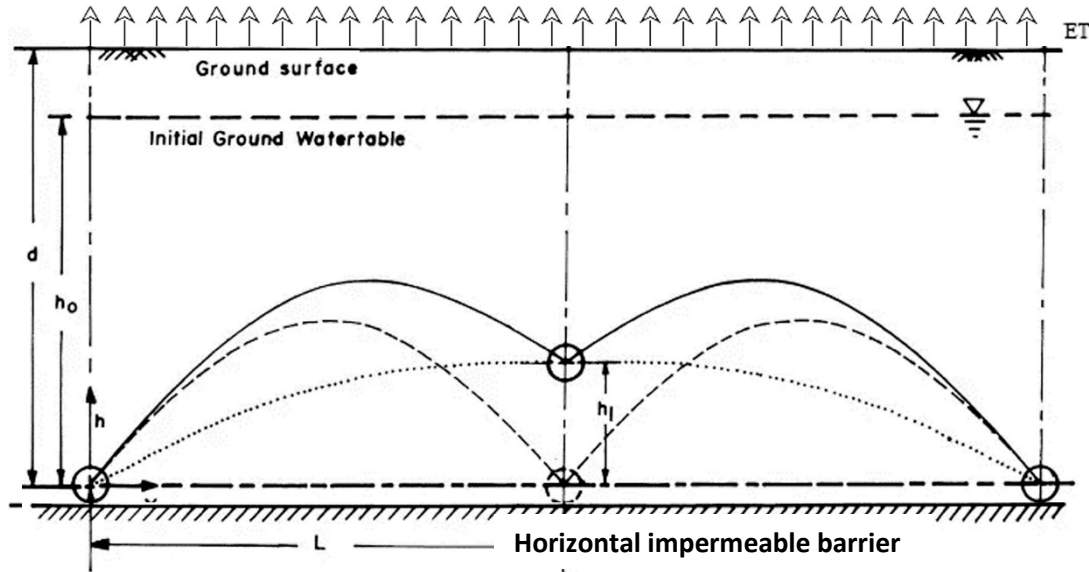


Figure 1 Schematic diagram of conventional and bi-level drainage problem

2.2 Governing equation, initial and boundary conditions

1D nonlinear BE incorporating depth dependent ET as sink term is given in Equation (1).

$$h \frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x} \right)^2 - \left\{ \frac{E_0 - b(h_0 - h)}{K} \right\} = \frac{f}{K} \frac{\partial h}{\partial t} \quad (1)$$

Where,

h is the height of water table [L] at any time t above the deep drain;

E_0 the rate of ET [LT^{-1}] at the land surface;

b the depth-dependent reduction factor [T^{-1}];

h_0 is the height of initial water table [L] close to the soil surface;

K is the average hydraulic conductivity [LT^{-1}];

f is the drainable porosity (dimensionless).

Linearized BE after employing Boumann's transformation, in which $(\partial h / \partial x)^2$ is neglected and h associated with $(\partial^2 h / \partial x^2)$ is replaced by average depth of flow 'D' is described in Equation (2).

$$\frac{\partial^2 h}{\partial x^2} - \left\{ \frac{E_0 - b(h_0 - h)}{KD} \right\} = \frac{f}{KD} \frac{\partial h}{\partial t} \quad (2)$$

2.3 Numerical and analytical solutions

In this section numerical solutions of nonlinear BE and analytical solution of linearized BE have been presented.

2.3.1 Hybrid finite analytic solution

First Equation (2) was non-dimensionalized with the help of a set of variables, $H = h/h_0$, $X = x/L$, and $T = Kh_0 t / L^2$. Transforming Equation 2 with these non-dimensional variables, the non-dimensional governing equation and IBC become as given in Equation (3).

$$H \frac{\partial^2 H}{\partial X^2} + \left(\frac{\partial H}{\partial X} \right)^2 - \left(\frac{E_0 - bh_0(1-H)}{Kh_0^2} \right) L^2 = \frac{\partial H}{\partial T} \quad (3)$$

HFAS of nonlinear non-dimensionalized Equation 3 along with IBC is obtained based on the approach of Chen (1988) for falling WT between two bi-level drains. A transformation is devised to absorb the terms of the depth dependent ET is represented as given in Equation (4).

$$H = V e^{-\frac{bL^2T}{Kh_0}} - \frac{E_0}{bh_0} + 1 \quad (4)$$

This transformation converts Equations (3) and (4) in the following form as given in Equation (5).

$$\frac{\partial^2 V}{\partial X^2} + \frac{1}{H_a} \left(\frac{\partial V}{\partial X} \right)^2 e^{-\frac{bL^2T}{Kh_0}} = \frac{1}{H_a} \frac{\partial V}{\partial T} \quad (5)$$

Where,

H_a is the dimensionless average depth of flow.

Assuming the term $\frac{1}{H_a} \left(\frac{\partial V}{\partial X} \right)^2 e^{-\frac{bL^2T}{Kh_0}}$

and $\frac{1}{H_a} \frac{\partial v}{\partial T}$ as constants denoted by C and E , respectively in a small sub-region and performing integration given in Equation (6).

$$\frac{dv}{dx} = (E - C)X + F \quad (6)$$

Again, by applying inverse transformation in Equation (4), the value of H at any space and time is obtained. The WT height (h) at any distance and time is obtained by multiplying H with h_0 .

2.3.2 Finite difference solution

Finite difference solution of nonlinear, non-dimensionalized BE as shown in Equation (3) to describe WT rise in an unconfined aquifer between two bi-level drains was obtained by using Galerkin's method, the details of which are given by Pinder and Gray (1977). Flow domain is discretized as $0 = X_1 < X_2 < X_3 < X_4 < \dots < X_{N-1} < X_N = 1$ (Where, N denotes to number of nodes. $\Delta X = X_{i+1} - X_i$, where $i = 1, 2, 3, \dots, N-1 = M$, the number of elements). The solution was approximated by $H^A(X, T)$ with the help of the basic functions is represented as given in Equation (7).

$$H^A(X, T) = \sum_{i=1}^N Z_i(T) \cdot N_i(X) \quad (7)$$

Where,

$Z_i(T)$ is unknown coefficients to be determined as a part of the solution and basis function;

$N_i(X)$, $N_{i-1}(X)$ and $N_{i+1}(X)$ as defined by Powers (1972).

To carry out finite element analysis may be written as given in Equation (8).

$$L(H) = \frac{\partial}{\partial X} \left(H \frac{\partial H}{\partial X} \right) - \left[\frac{E_0 - bh_0(1-H)}{K h_0^2} \right] L^2 - \frac{\partial H}{\partial T} = 0 \quad (8)$$

2.3.3 Fully Implicit Finite Difference Solution

Finite difference solution of nonlinear non-dimensionalized BE (3) with IBC was obtained by discretizing in finite difference form as given in Equation (9).

$$\frac{H_m^{n+1} - H_m^n}{\Delta T} = \frac{1}{2(\Delta X)^2} [\theta(H_{m-1}^{n+1})^2 + (1-\theta)(H_{m-1}^n)^2 - 2\theta(H_m^{n+1})^2 - 2(1-\theta)(H_m^n)^2 + \theta(H_{m+1}^{n+1})^2 + (1-\theta)(H_{m+1}^n)^2] - \left[\frac{E_0 - bh_0\{1-\theta(H_m^{n+1}) - (1-\theta)H_m^n\}}{K h_0^2} \right] L^2 \quad (9)$$

Where,

θ is coefficient used to describe the finite difference scheme. The 1, 0.5 and 0 values of θ describe fully implicit, Crank-Nicolson, and fully

explicit finite difference solution.

2.3.4 Analytical solution I

Analytical solution of BE linearized by baumann's transformation (Equation 2) with IBC was developed by devising a transformation is represented given in Equation (10).

$$h = v e^{-\frac{bt}{f}} - \frac{E_0}{b} + h_0 \quad (10)$$

Using this transformation, Equation (2) is transformed to a standard heat transfer boundary value problem as given in Equation (11).

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{a} \frac{\partial v}{\partial t} \quad (11)$$

Here, $a = KD/f$ with IBC as:

Employing the solution of heat transfer boundary value problem given by Ozisik (1980) for the boundary value problem defined by Equation (11) and applying inverse of transformation solution is obtained and presented as given in Equation (12).

$$h(x, t) = \frac{4}{L} e^{-\frac{bt}{f}} \sum_{m=1,3,5,\dots}^{\infty} \frac{\sin \beta_m x}{\beta_m} \left[h_0 e^{-a\beta_m^2 t} - \frac{1}{f} (E_0 - bh_0) \left(\frac{e^{\frac{bt}{f}} - e^{-a\beta_m^2 t}}{\frac{b}{f} + a\beta_m^2} \right) \right] + \frac{2}{L} e^{-\frac{bt}{f}} \sum_{m=1}^{\infty} (-1)^m \frac{\sin \beta_m x}{\beta_m} \left[h_1 e^{-a\beta_m^2 t} + \frac{bh_1}{f} \left(\frac{e^{\frac{bt}{f}} - e^{-a\beta_m^2 t}}{\frac{b}{f} + a\beta_m^2} \right) \right] + \frac{X}{L} h_1 \quad (12)$$

To compare WT profiles obtained from analytical and numerical solutions with WT obtained from HFAS, L2 norm giving average difference and Tchebycheff norm giving maximum difference were employed and accordingly results were interpreted. Similarly, spacing between drains as obtained by various solutions was compared with that obtained from HFAS and percentage change in spacing was determined for a selected numerical example.

3 Results and discussion

To compute lowering WT between two drains from initial flat water table near the ground surface due to instantaneous recharge, drains are assumed to be separated by 50 m. Deep drain is lying at horizontal impermeable barrier at the depth of 1.8 m from the ground surface, whereas shallow drain is assumed to

be located at 1.2 m bgl or 0.6 m above the deep drain level. The hydraulic conductivity (K) and specific yield (S_y) of the soil is 3.0 m day^{-1} and 0.14, respectively. The E_0 is assumed as $8 \times 10^{-3} \text{ m day}^{-1}$ and b is 6.67×10^{-3} per day. In numerical solutions time increment (ΔT), and space increment (ΔX) were considered as 1×10^{-6} and 0.05, respectively. The transient lowering of WT has been determined after 1, 2, 3, and 4 days, and spacing considering unsteady state drainage criteria have been determined.

3.1 Falling water tables between two bi-level drains

as obtained from HFAS, FES, FIFDS and AS

Transient lowering of WT between two bi-level drains for the selected example was investigated by applying HFAS, FES, FIFDS and AS, and influence of constant ET and depth dependent ET as compared to no ET on lowering of WT was studied. Since assumption of constant ET throughout the soil profile does not seem to be realistic so transient WT considering zero ET and depth dependent ET as obtained from HFAS, FES, FIFDS and AS have been presented in Figures 2, 3, 4, and 5, respectively.

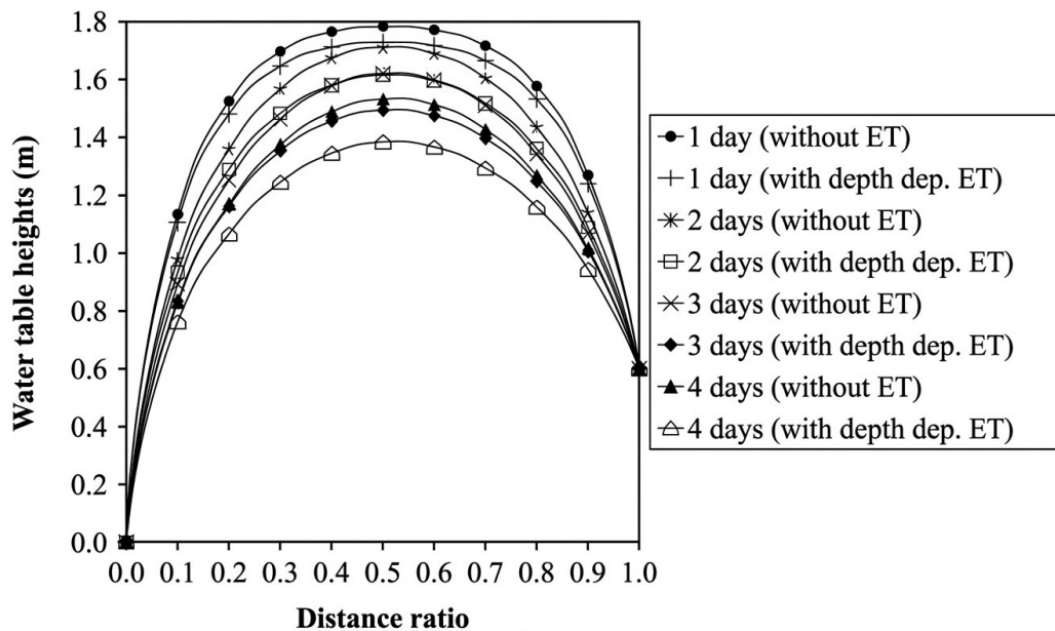


Figure 2 Transient lowering of water table between two bi-level drains in a horizontal aquifer obtained from HFAS

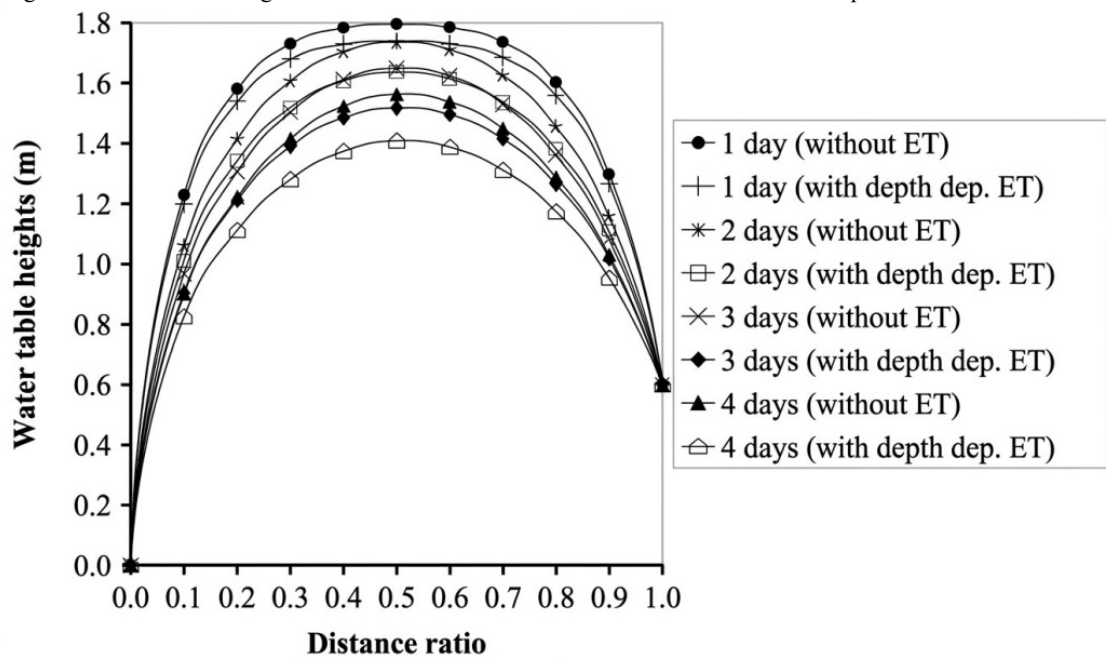


Figure 3 Transient lowering of water table between two bi-level drains in a horizontal aquifer obtained from FES

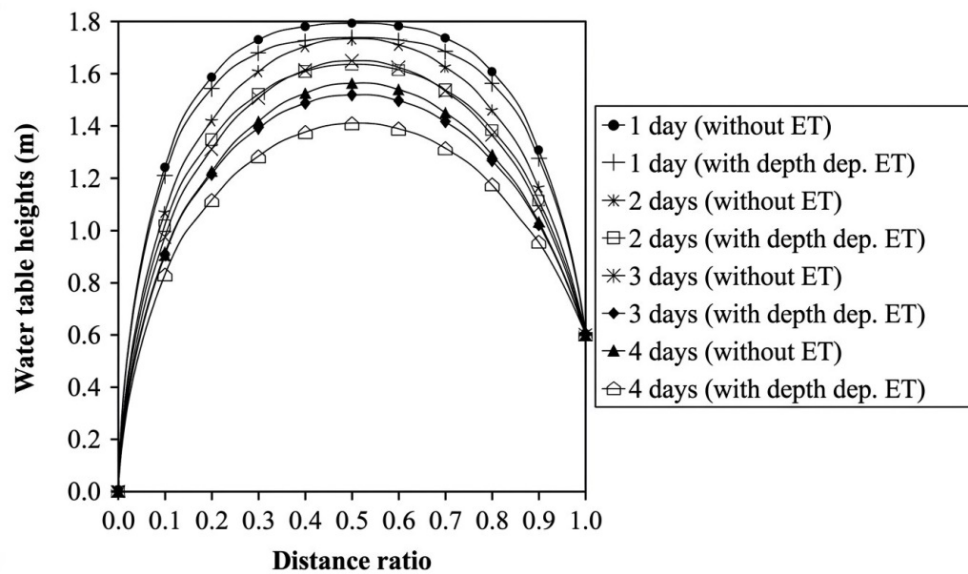


Figure 4 Transient lowering of water table between two bi-level drains in a horizontal aquifer obtained from FIFDS

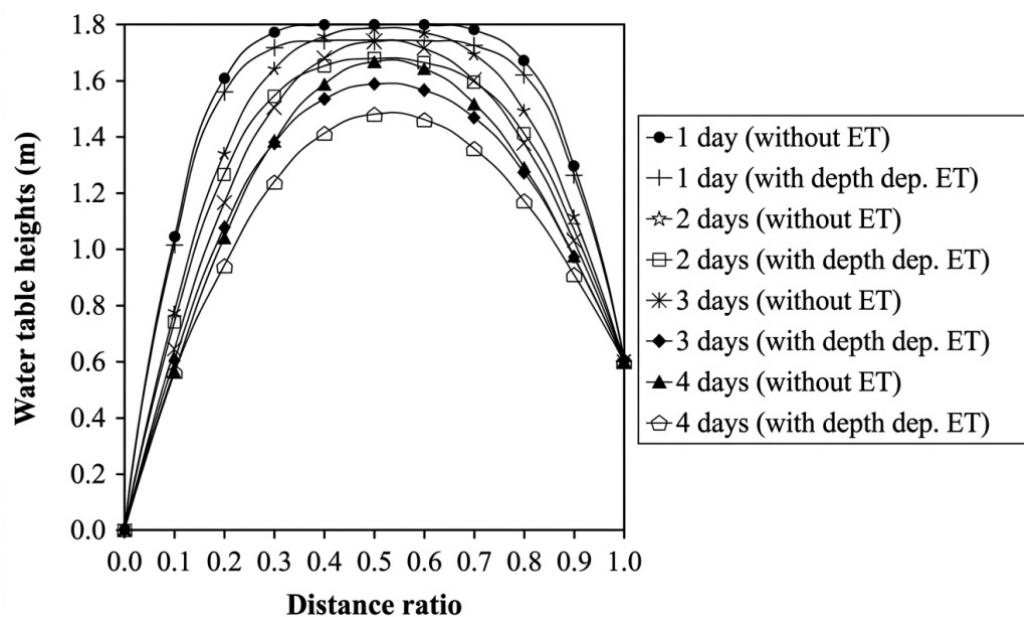


Figure 5 Transient lowering of water table between two bi-level drains in a horizontal aquifer obtained from AS based on Baumann's linearization method

It may be observed from Figures 2, 3, 4 and 5 that due to depth dependent ET, lowering of WT is relatively faster as compared to zero ET. If a constant ET is assumed to occur inside the soil profile, the lowering of WT is found to be the fastest one. Figures 2 to 5 show that the lowering of WT in a horizontal aquifer is symmetrical with maximum WT elevation occurring at mid-point. The difference in maximum WT elevations is due to zero ET and depth dependent ET increases with increase in time. Analytical solutions (AS) for water table (WT) fluctuations derived from the linearized Boussinesq equation

typically underestimate water table rise and fall, because they neglect nonlinear, large-gradient effects that become important during drainage processes.

3.2 Comparison of water table elevations obtained by HFAS with FES, FIFDS and AS

Water table elevations at 1, 2, 3, and 4-day as obtained from HFAS were compared with the EFS, FIFDS and AS to study the difference in values of WT elevations considering HFAS as the benchmark solution. Lowering WT obtained by these solutions considering zero ET and depth dependent ET at 2-day and 3-day have been presented in Figures 6, 7 and 8.

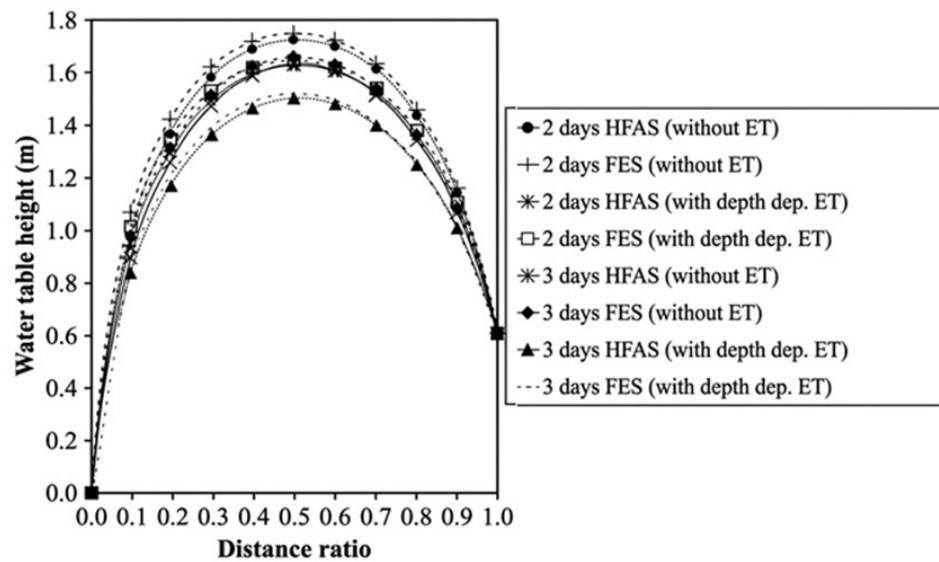


Figure 6 Water table on 2-day and 3-day between two bi-level drains in a horizontal aquifer obtained from HFAS and FES

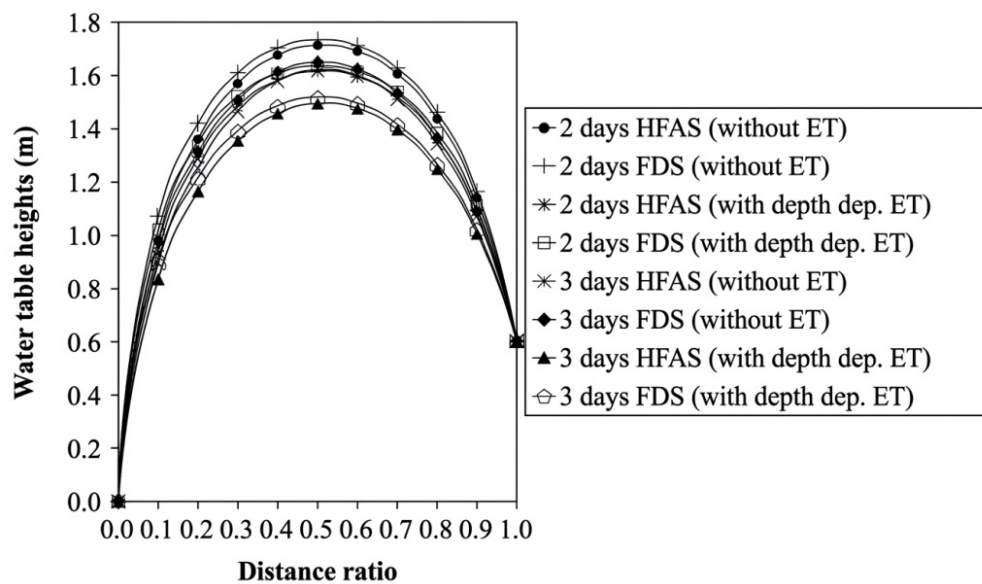


Figure 7 Water table on 2-day and 3-day between two bi-level drains in a horizontal aquifer obtained from HFAS and FIFDS

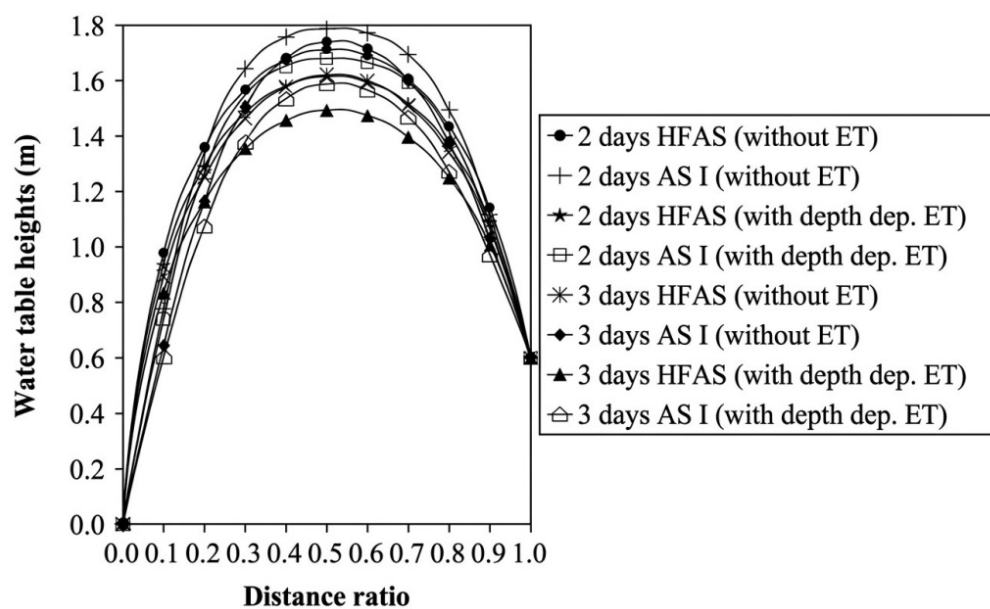


Figure 8 Water table on 2-day and 3-day between two bi-level drains in a horizontal aquifer obtained from HFAS and AS

It may be observed from Figures 6, 7, and 8 that mid-point WT elevations as obtained from HFAS on 2-day and 3-day are lower than those obtained from FES, FIFDS and maximum difference when compared with WT elevations obtained from AS.

3.3 L2 and Tchebycheff norms

Table 1 L2 and Tchebycheff norms to compare transient falling water table between two bi-level drains on 1, 2, 3, and 4-day obtained from FES, FIFDS and AS with HFAS

Norms	FES	FIFDS	AS	FES	FIFDS	AS
	Water table fall (m) without considering ET			Water table fall (m) considering depth-dependent ET		
1-day						
L2	0.0402	0.0441	0.0607	0.0390	0.0427	0.0574
Tchebycheff	0.0960	0.1070	0.1250	0.0930	0.1040	0.1099
2-day						
L2	0.0382	0.0407	0.0885	0.0360	0.0381	0.0807
Tchebycheff	0.0830	0.0900	0.2030	0.0790	0.0850	0.1940
3-day						
L2	0.0374	0.0396	0.1095	0.0341	0.0356	0.0960
Tchebycheff	0.0760	0.0810	0.2480	0.0710	0.0750	0.2310
4-day						
L2	0.0369	0.0384	0.1199	0.0324	0.0338	0.0915
Tchebycheff	0.0720	0.0750	0.2670	0.0650	0.0680	0.1990

It may be observed from Table 1 that in case of bi-level drainage system, the values of L2 and Tchebycheff norms for both the cases of zero ET and depth dependent ET on 1, 2, 3 and 4-day are minimum for FES, with successively increasing values, respectively, for FIFDS and maximum for AS. It may also be observed from the above that the difference in the values of L2 and Tchebycheff norms for FES and FIFDS is negligible, which indicates that performance of both the solutions with respect to HFAS may be treated as equally good and practically acceptable. The values of L2 and Tchebycheff norms for AS are more than both numerical solutions. It indicates that behavior of AS changes due to the linearization of a nonlinear partial differential equation governing the flow phenomenon.

3.4 Spacing between two bi-level drains computed by HFAS, FES, FIFDS and AS

Using the same parameters except drain spacing as given in the numerical example, spacing between two bi-level drains was computed employing the implicit HFAS for an assumed drainage criterion of lower of maximum WT by 30 cm in 2 days from an initially flat

In order to rank the theoretical performance of various analytical and numerical solutions with respect to hybrid finite analytic solution, L2 and Tchebycheff norms indicating average and maximum difference from the benchmark solution were computed and are presented in Table 1 for 1, 2, 3 and 4-day, respectively.

WT near the soil surface and results are presented in Table 2.

The drain spacing between bi-level drains was obtained for the selected example considering $H_0 = 1.8$ m, $H_1 = 0.6$ m, $H_r = 1.5$ m, $K = 3.0$ m/d, $f = 0.14$, $E_0 = 0.008$ m/d, $b = 0.00667$, $t = 2$ days by applying HFAS, FES, FIFDS and AS. It may be observed from Table 2 that spacing between two parallel drains as obtained from all the four solutions increases in the range of 10.27% (in case of AS) to 13.16% (in case of HFAS), when depth dependent ET is considered as compared to no ET.

Spacing computed by various solutions both for the cases of no ET and depth dependent ET were compared. Spacing for the case of no ET and depth dependent ET computed by FES, FIFDS, and AS is 4.06 % and 4.58%; 4.18% and 4.66%; 14.43% and 16.51%; less than computed by HFAS.

3.5 Effect of variation of rates of ET on spacing between two bi-level drains as obtained from HFAS, FES, FIFDS and AS

The effect of various rates of ET on spacing between two bi-level drains lying on horizontal

impermeable barrier was studied and is presented in Table 3.

Table 2 Computation of spacing (m) between bi-level drains obtained from HFAS, FES, FIFDS and AS

Condition	HFAS	FES	FIFDS	AS
Spacing without ET	33.74	32.37	32.33	28.91
% decrease in spacing compared to HFAS	-	4.06	4.18	14.43
Spacing with depth-dependent ET	38.18	36.43	36.40	31.88
% decrease in spacing compared to HFAS	-	(4.58)	(4.66)	(16.51)
% increase in Spacing considering depth dependent ET compared to no ET	13.16	12.54	12.59	10.27

Table 3 Effect of various rates of E_0 on computed spacing (m) between two bi-level drains using HFAS, FES, FIFDS and AS

Values of E_0 (m d^{-1})	HFAS	FES	FIFDS	AS
Spacing with $E_0 = 0.0$	33.74	32.37	32.33	28.91
% decrease in spacing compared to HFAS	-	4.06	4.18	16.71
Spacing with $E_0 = 0.004$	35.25	33.74	33.73	29.93
% decrease in spacing compared to HFAS	-	4.28	4.31	15.09
% increase in spacing with $E_0 = 0.004$ compared to zero E_0	4.48	4.23	4.33	3.53
Spacing with $E_0 = 0.006$	36.62	35.01	34.98	30.86
% decrease in spacing compared to HFAS	-	4.40	4.48	15.73
% increase in spacing with $E_0 = 0.006$ compared to zero E_0	8.54	8.16	8.20	6.75
Spacing with $E_0 = 0.008$	38.18	36.43	36.40	31.88
% decrease in spacing compared to HFAS	-	4.58	4.66	16.50
% increase in spacing with $E_0 = 0.008$ compared to zero E_0	13.16	12.54	12.59	10.27

Table 4 Effect of parameter 'b' on computed spacing (m) between bi-level drains using HFAS, FES, FIFDS and AS

Values of b	HFAS	FES	FIFDS	AS
Spacing with $b = 0$	39.36	37.50	37.45	32.62
% decrease in spacing compared to HFAS	-	4.73	4.85	17.12
Spacing with $b = 0.0055$	38.39	36.62	36.58	32.01
% decrease in spacing with $b = 0.0055$ compared to $b = 0$	2.46	2.35	2.32	1.87
% decrease in spacing compared to HFAS	-	4.61	4.71	16.62
Spacing with $b = 0.00667$	38.18	36.43	36.40	31.88
% decrease in spacing with $b = 0.00667$ compared to $b = 0$	3.00	2.85	2.80	2.27
% decrease in spacing compared to HFAS	-	4.58	4.66	16.50
Spacing with $b = 0.00778$	37.99	36.23	36.22	31.74
% decrease in spacing with $b = 0.00778$ compared to $b = 0$	3.48	3.39	3.28	2.70
% decrease in spacing compared to HFAS	-	4.63	4.66	16.45
Spacing with $b = 0.0088$	37.79	36.08	36.07	31.62
% decrease in spacing with $b = 0.0088$ compared to $b = 0$	3.99	3.79	3.68	3.07
% decrease in spacing compared to HFAS	-	4.53	4.55	16.33

It may be observed from Table 3 that for the rate of ET varying from 0.004 m day^{-1} to $8 \times 10^{-3} \text{ m day}^{-1}$ spacing between two bi level drains increases from 3.53 % (in AS) to 13.16% (in HFAS) as compared to the case of zero ET. Comparison of spacing computed by FES, FIFDS and AS compared to HFAS for the cases of ET varying from zero to $8 \times 10^{-3} \text{ m day}^{-1}$ showed decrease in spacing from 4.06% to 4.58% in

FES; 4.18% to 4.66% in FIFDS; and 15.09% to 16.71% in AS, respectively.

3.6 Effect of parameter 'b' on spacing between two bi-level drains lying on horizontal impermeable barrier as in HFAS, FES, FIFDS and AS

Effect of depth dependent reduction factor 'b' on spacing between two bi-level drains lying on the horizontal / sloping impermeable barrier was studied

and is given in Table 4.

Table 4 shows that with increase in the value of b from 0 to 8.8×10^{-3} per day, the spacing between two drains decreases by 3.99% in HFAS; 3.79% in FES; 3.68% in FIFDS; and 3.07% in AS. When spacing value as obtained from HFAS was compared with other solutions considering value of b ranging from 0 to 8.8×10^{-3} per day, the percentage reduction in spacing was observed in the range of 4.53% to 4.73% for FES; 4.55 %

to 4.85 % for FIFDS and 16.33 % to 17.12 % for AS.

3.7 Field applications

Determinations of spacing between drains play an important role in design of a drainage system. If spacing is more, the design becomes economical because it needs lesser pipes to drain. Similarly, in bi-level drainage system such as alternate deep and shallow drains are laid, cost of excavation reduces substantially. Spacing between bi-level drains as obtained from AS was least and from HFAS, it was maximum. Hybrid finite analytic solution (HFAS) is simple and accurate as compared to other solutions, so its application in determination of design after considering depth dependent ET may be recommended for semi-arid and arid regions.

4 Conclusion

In order to compute spatial and temporal variations of the water table (WT) and drain spacing between bi-level drains, three numerical solutions (HFAS, FES, and FIFDS) of the 1D nonlinear Boussinesq equation and an analytical solution (AS) of the linearized equation with depth-dependent ET were used. Incorporating depth-dependent ET accelerated WT falling and increased drain spacing by AS to HFAS compared to the no-ET. Galerkin's method based finite element solution (FES) ranked first in predicting faster water table decline, followed by FIFDS, while AS ranked last. Increasing ET increased drain spacing relative to no ET, whereas increasing the depth-dependent reduction factor reduced spacing for FES, FIFDS, and AS. Solving the Boussinesq equation using HFAS leads to a more rapid decline of the water

table and, consequently, larger drain spacing. In HFAS, the nonlinear Boussinesq equation is locally linearized and analytically solved, preserving nonlinearity through the assembly of locally analytic solutions. It closely matches experimental midpoint water table data and, due to its simplicity and accuracy, is recommended for estimating water table falling and drain spacing under depth-dependent ET in semi-arid and arid regions.

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